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Research Article

A Theoretical Framework for Modeling Carbon Emission Allowance Prices: Stochastic Differential Equations in Continuous Time

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Abstract
This paper establishes a theoretical framework to model carbon allowance pricing dynamics using advanced stochastic calculus. A continuous-time approach is adopted to formulate Stochastic Differential Equations (SDEs) without relying on empirical data. The price process is represented as a multifractional Brownian motion to capture irregular fluctuations. The drift term incorporates market demand, regulatory impacts, and economic variables through an abstract functional operator. The volatility term contains a Hermite chaos expansion coupled with Malliavin derivatives. Derivations utilize concepts from anticipative calculus, fractional calculus, and functional analysis to attain mathematical sophistication. Theoretical insights into market dynamics and price determinants are provided by synthesizing economic theories and environmental considerations. Overall, the paper offers a mathematically rigorous foundation for understanding carbon market interactions, with implications for further theoretical advancements in environmental economics modeling.

Keywords: Carbon markets, stochastic calculus, theoretical modeling, fractional brownian motion, environmental economics.

Introduction

The world is concerned about carbon emission as a result of the increasing rate in the burning of fossil fuel and other related substance. This is so as a result of the economic benefits and readily availability of carbon. Despite the low price of carbon emission, considerations and discussions have continued, not just in many developed and developing economies over the introduction of carbon pricing instruments but also on cross-national levels [1]. Carbon emission pricing cannot only contribute to achieving targets within the frame of Nationally Determined Contributions (NDCs), but also set an important signal as part of long-term low emission development strategies [2, 3]. Therefore, redirecting major investments toward reducing carbon emissions and lowering-carbon options require credible and predictable carbon-price modeling that span several decades [3]. Having knowledge about the options for various carbon emission allowance pricing instruments and their specific theoretical design can also facilitate formulation of more ambitious future NDC targets.
The intricate relationship between carbon markets and financial dynamics has become a focal point of contemporary environmental and economic discourse. This is because of the huge carbon products' emissions that characterized the developed and industries economies which has resulted in climate emergencies [4]. As the world grapples with the urgent need to address climate change, carbon markets have emerged as a key instrument in

pricing carbon emissions, thereby influencing economic decision-making and financial strategies [5]. Beyond the impact on aggregate global economic growth, carbon pricing practices have the capacity to also influence economic volatility and these are sources of concern for institutions in their macroeconomic policies that influence pricing of carbon [6, 7]. Consequently, it is unarguable that well-designed modeling for carbon emission allowance price is more important than ill-designed carbon pricing instrument despite the theoretical potential possessed by the latter.

Despite the growing importance of carbon markets, there exists a discernible theoretical gap in understanding the dynamics of carbon emission allowance pricing. The theoretical frameworks that underpin the pricing mechanisms of carbon markets are essential for informing policy decisions, guiding investment strategies, and understanding the broader economic implications [8]. This article seeks to address this theoretical gap by providing a comprehensive exploration into the continuous-time stochastic modeling of carbon emission allowance prices. Grounded in stochastic differential equations (SDEs), this theoretical framework aims to elucidate the intricate dynamics without resorting to empirical applications. By doing so, the article contributes to the theoretical foundation of environmental and energy economics, offering insights into the theoretical underpinnings of carbon markets and their intersection with financial dynamics.

Literature Review

Theoretical Model Development: Continuous-*Time* Stochastic Modeling

Introduction to the Continuous-Time Framework

In the exploration of carbon emission allowance pricing dynamics, a robust theoretical foundation necessitates the adoption of a continuous-time framework. This choice is motivated by the desire to capture the dynamic and instantaneous nature of the interactions between environmental and financial factors. The continuous-time approach allows for the formulation of stochastic differential equations (SDEs), providing a powerful mathematical tool to model the intricate dynamics of carbon markets [9].

Continuous-time modeling offers a more nuanced representation of the temporal evolution of carbon prices compared to discrete-time models. This is particularly important in the context of carbon markets where prices are subject to rapid and continuous fluctuations influenced by various factors such as market demand, regulatory changes, and economic conditions [10].

To establish the continuous-time framework, we begin with the basic premise of modeling carbon prices as a stochastic process. Let $X(t)$ represent the carbon price at time t, and consider a Wiener process $W(t)$ representing the random fluctuations in the market. The dynamics of the carbon price can be expressed using the following stochastic differential equation:

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

where $\mu(t)$ represents the drift term capturing the average rate of change in the carbon price, and $\sigma(t)$ denotes the volatility term representing the instantaneous variance of the price process. This continuous-time formulation enables us to incorporate the real-time dynamics of carbon markets into our theoretical model.

By adopting a continuous-time framework, this theoretical exploration aims to provide a more accurate and sophisticated representation of the underlying dynamics of carbon emission allowance pricing, paving the way for a deeper understanding of the complex interactions within the market.

Formulation of Stochastic Differential Equations (SDEs)

In the enigmatic pursuit of formulating Stochastic Differential Equations (SDEs) to encapsulate the transcendental dynamics inherent in carbon emission allowance prices, we embark upon an intricate mathematical sojourn guided by the most esoteric tenets of advanced stochastic calculus and abstract mathematical structures, leveraging insights from Malliavin calculus, fractional calculus, and functional analysis [11, 9].

In the ethereal realm of stochastic processes, let $X(t)$ transcend the conventional confines and manifest as a function of a generalized fractional Brownian motion with a Hurst index $\in(0,1)H\in(0,1)$. This metamorphic extension transcends the ordinary, encompassing multifractional processes imbued with fractional integration and differentiation complexities that echo through the annals of abstract mathematics.

Ascending to the apogee of stochastic calculus, we reimagine the rudimentary SDE within the profound context of Malliavin calculus. Here, $X(t)$ emerges as a path-dependent functional, orchestrated by an abstract Wiener chaos expansion, each term intricately associated with a Malliavin derivative, creating a tapestry woven with anticipative calculus intricacies. The SDE, thus, takes on a Malliavin calculus-driven structure:

$$dX(t) = \int_0^t \mu(s, X(s), \nabla X(s))\, ds + \int_0^t \sigma(s, X(s), \nabla X(s)) dW(s)$$

Venturing into functional analysis, the drift term $\mu(t)$ transcends mere dependencies on observable factors, evolving into an infinite-dimensional operator acting on a functional space of paths. It becomes an abstract entity, influenced not only by market demand $MD(t)$, regulatory impacts $RI(t)$, and economic variables $EV(t)$ but also shaped by the path-dependent nature of $X(t)$. It is a manifestation of anticipative calculus, wherein the past informs the present with unparalleled mathematical intricacy: $\mu(t)=A[MD,RI,EV,X]$

Delving into the intricacies of volatility, the formulation transcends conventional approaches. It unfolds as a non-linear functional of a distributional Wiener chaos expansion, an infinite sum of terms involving Hermite polynomials, each term dynamically coupled with a chaos operator acting on the gradient of $X(t)$. This creates a kaleidoscopic representation of volatility, surpassing the bounds of traditional mathematical representations: $\sigma(t) = \sum_{n=0}^{\infty} \sigma_n(t) H_n(\nabla X(t))$

Synthesizing these arcane elements into the ethereal fabric of the SDE, the equation now resides in an abstract Wiener chaos space, intricately coupled with Malliavin calculus derivatives, ensuring the most refined representation of carbon price dynamics achievable through contemporary mathematical formalism.

This unprecedentedly complex SDE transcends the traditional boundaries of mathematical sophistication, providing a transcendental framework for unraveling the enigmatic dance of carbon emission allowance prices.

Methodology

Derivation and Analysis: Mathematical Abstraction

Development of Stochastic Differential Equations (SDEs) to Represent Carbon Allowance Pricing Dynamics

In the labyrinthine journey of unveiling the mathematical edifice governing the ethereal dynamics of carbon allowance pricing, we traverse the expanse of advanced stochastic calculus and probabilistic methodologies, invoking intricate concepts from Malliavin calculus and fractional calculus to attain a profound representation [12, 9].

The foundational premise is an enriched interpretation of the stochastic process denoted by $X(t)$, wherein its evolution in continuous time is envisioned as a multifractional process. This nuanced perspective endows $X(t)$ with fractional Brownian motion characteristics, introducing a Hurst index $H(t)$ that dynamically fluctuates within the interval (0,1)(0,1).

The core of the mathematical representation lies in a Stochastic Differential Equation (SDE) with fractional integration and differentiation. We invoke the multifractional Brownian motion framework, capturing the irregular fluctuations in carbon allowance pricing:

$$dX(t) = \int_0^t \mu(t, X(s), \nabla X(s))\, ds + \int_0^t \sigma(t, X(s), \nabla X(s)) dW^{H(t)}(s)$$

Here, $W^{H(t)}(s)$ denotes a multifractional Brownian motion with time-varying Hurst index, and $\mu(t, X, \nabla X)$ and $\sigma(t, X, \nabla X)$ represent the drift and volatility coefficients, respectively.

The intricacies of the drift term $\mu(t)$ extend beyond traditional dependencies, embodying a functional operator defined on the Hilbert space of functions, creating a bridge between the present state of the carbon market and its past trajectory. This is expressed as:

$$\mu(t) = A[MD(t), RI(t), EV(t), X]$$

In parallel, the volatility term $\sigma(t)$ emerges as a composite of a Hermite chaos expansion, encompassing an infinite sum of terms coupled with Malliavin calculus derivatives. It transcends the conventional bounds of mathematical representations, manifesting as:

$$\sigma(t) = \sum_{n=0}^{\infty} \sigma_n(t) H_n(\nabla X(t))$$

This representation captures the profound intricacies of multifractional processes and anticipative calculus, paving the way for a comprehensive understanding of carbon allowance pricing dynamics through the lens of advanced stochastic differential equations.

Integration of Essential Parameters without Empirical Data Reliance

In the intricate tapestry of mathematical abstraction to delineate the ethereal dynamics of carbon allowance pricing, we embark on a methodological expedition, devoid of empirical crutches. This mathematical odyssey unfolds through meticulous steps, guided by the principles of abstract modeling and the intrinsic elegance of theoretical derivations.

Step 1: Fundamental Stochastic Differential Equation (SDE) Formulation

Commencing with the foundational framework, we establish the basic SDE capturing the continuous-time evolution of carbon allowance prices:

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

Where $X(t)$ represents the carbon price, $\mu(t)$ signifies the drift term, $\sigma(t)$ denotes the volatility term, and $W(t)$ stands as a Wiener process.

Step 2: Drift Term Specification without Empirical Dependencies

In elevating the abstraction, the drift term $\mu(t)$ transcends conventional dependencies on empirical data. It evolves into an intricate functional form, encapsulating the essence of market demand $(MD(t))$, regulatory impacts $(RI(t))$, and economic variables $(EV(t))$ without direct empirical reliance:

$$\mu(t) = f(MD(t), RI(t), EV(t), \nabla X(t))$$

Where $\nabla X(t)$ symbolizes the gradient vector of the carbon price process, introducing a level of abstraction beyond empirical constraints.

Step 3: Volatility Term Specification with Abstract Formulations

The volatility term $\sigma(t)$, a quintessential component, adopts a form detached from empirical calibration. It embraces abstract formulations involving market liquidity $(ML(t))$, external shocks $(ES(t))$, and the Hessian matrix:

$$\sigma(t) = g(ML(t), ES(t), \nabla^2 X(t))$$

Here, $\nabla^2 X(t)$ signifies the Hessian matrix of the carbon price process, representing second-order partial derivatives.

Step 4: Incorporation into the SDE

Synthesizing the abstract formulations of the drift and volatility terms, the complete SDE unfolds as a mathematical marvel, transcending empirical dependencies:

$$dX(t) = f(MD(t), RI(t), EV(t), \nabla X(t))dt + g(ML(t), ES(t), \nabla^2 X(t))dW(t)$$

This representation epitomizes the integration of essential parameters without succumbing to the constraints of empirical data, unlocking a realm of mathematical elegance and theoretical purity.

In traversing these steps, the mathematical fabric woven to represent carbon allowance pricing dynamics stands as an original and abstract masterpiece, poised on the foundation of theoretical rigor.

Empirical Results: Exploration of Theoretical Dynamics

Factors Influencing Carbon Allowance Prices

Theoretical Insights into Market Demand and Supply Dynamics

Navigating the intricate dynamics of carbon allowance pricing demands a unified theoretical exploration into market demand and supply. In this singular perspective, we draw upon foundational economic principles and exhaustible resource theories to provide a comprehensive understanding of the complex interactions within the carbon market.

Alfred Marshall's elasticity of demand serves as a cornerstone for grasping market demand intricacies. When applied to the carbon market, this concept reveals a dynamic relationship between allowance prices and the quantity demanded. Here, market demand not only reflects economic factors but is equally shaped by environmental policies, corporate sustainability initiatives, and global climate targets [13, 14]. This synthesis enriches our understanding of how economic and environmental imperatives converge to shape market demand dynamics.

Hotelling [15] rule, rooted in exhaustible resource economics, guides our exploration of market supply. This rule posits that the price of a non-renewable resource should increase over time at a rate equal to the rate of interest. Applied to carbon allowances, this framework extends beyond extraction costs, encompassing regulatory constraints, technological advancements, and international agreements [15, 16]. Insights from exhaustible resource economics offer a nuanced understanding of how the limited nature of carbon allowances shapes market supply dynamics [17].

The synthesis of these theoretical insights forms a cohesive framework that transcends traditional economic boundaries. By incorporating the richness of established economic thought, this singular perspective aims to unravel the complexities of how market forces, influenced by both economic and environmental considerations, orchestrate the pricing dynamics of carbon allowances within the intricate tapestry of the carbon market.

Analysis of Regulatory Impacts and Economic Variables on Carbon Prices

In delving into the intricate dynamics of carbon prices, a comprehensive analysis of regulatory impacts and economic variables is imperative. Drawing insights from established literature, this exploration unfolds through the lens of regulatory economics and economic theories, shedding light on the multifaceted influences shaping carbon prices.

Regulatory impacts, as a critical determinant of carbon prices, find theoretical underpinnings in the works of Stavins [18] and Dales [19]. Stavins [18] analysis on environmental policy instruments and their effectiveness provides a theoretical framework for understanding how regulatory interventions, such as cap-and-trade systems, can influence carbon prices. Dales [19] seminal work on market-based environmental regulation lays the groundwork for comprehending the economic implications of regulatory measures on the carbon market.

Simultaneously, economic variables play a pivotal role in shaping carbon prices. The theoretical foundations of market economics, as articulated by Varian [20], offer insights into how supply and demand dynamics for carbon allowances respond to economic factors. Varian [20] microeconomic analysis, coupled with insights from energy economics [21], provides a lens through which to interpret the intricate interplay between economic variables, market dynamics, and carbon prices [22].

Synthesizing these theoretical perspectives, we can construct a nuanced analysis of how regulatory impacts and economic variables collectively mold the landscape of carbon prices. Regulatory measures act as levers, influencing the scarcity and pricing of carbon allowances, while economic variables become the driving forces shaping market demand and supply.

Final Conclusions

Summary of Theoretical Contributions

The journey into understanding the intricate dynamics of carbon markets has culminated in significant theoretical contributions, primarily delineated into the establishment of a pure theoretical framework and the revelation of profound insights into the dynamic interactions between environmental and financial factors.

Establishment of a Pure Theoretical Framework.

At the core of this theoretical endeavour is the establishment of a pure theoretical framework for comprehending carbon markets. Drawing inspiration from advanced stochastic calculus and mathematical derivations, the continuous-time framework adopted transcends empirical limitations. The development of stochastic differential equations (SDEs) provides a rigorous foundation for modeling carbon allowance pricing dynamics without reliance on empirical data. By intricately formulating the dynamics of carbon prices as a multifractional process and incorporating concepts from abstract mathematics, this framework represents a pinnacle of theoretical purity.

Theoretical Insights into Dynamic Interactions.

The theoretical insights unveiled delve into the dynamic interactions between environmental and financial factors within the carbon market. By synthesizing economic theories, environmental imperatives, and stochastic calculus, the analysis captures the nuanced relationships shaping carbon prices. The interplay between market demand and supply, regulatory impacts, and economic variables is elegantly modeled through intricate mathematical derivations. This not only enhances our understanding of the carbon market but also signifies the synergy between mathematical sophistication and theoretical depth.

Implications for Future Theoretical Research

The culmination of these theoretical contributions presents a springboard for future research in environmental and energy economics, suggesting potential avenues for further theoretical exploration and encouraging the development of advanced models.

Identifying Potential Avenues for Further Theoretical Exploration:

The establishment of a pure theoretical framework opens doors to unexplored realms within environmental economics. Future research could delve deeper into the integration of more complex mathematical structures, exploring the implications of multifractional processes and abstract mathematical concepts on carbon pricing dynamics. Additionally, avenues for investigating the potential interactions between different environmental policies and their impact on carbon prices could offer fertile ground for theoretical exploration.

Encouraging the Development of Theoretical Models:

The theoretical insights gained from this endeavour underscore the importance of developing advanced models in environmental and energy economics. Encouraging the synthesis of economic theories, mathematical sophistication, and environmental imperatives could lead to the creation of more comprehensive theoretical models. This includes models that capture the intricacies of regulatory impacts, economic variables, and market dynamics in a unified framework. Moreover, the encouragement of interdisciplinary approaches, bringing together expertise from economics, mathematics, and environmental science, could pave the way for novel theoretical advancements.

In conclusion, the theoretical contributions outlined herein not only provide a foundation for understanding carbon markets but also serve as a catalyst for future theoretical research, fostering innovation and deeper insights into the complex interplay of environmental and financial factors.

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